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OPTIMAL CONTROL FOR A CLASS OF SYSTEMS SUBJECTED TO DISTURBANCES†

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A special class of linear dynamical systems acted upon by controls and bounded disturbances is considered. It is assumed that the disturbances are due to an error in the control implementation; there are no disturbances in the case of the zero control, but the range of the disturbances increases with the control intensity. The problem of constructing a control which provides a minimax to a specified optimality criterion for an arbitrary permissible form of the disturbances is formulated. The solution of the minimax problem is reduced to the solution of transcendental equations. Under certain conditions the solution is obtained in explicit form. An example is considered in which the optimal control is constructed both in the form of a program and in the form of a synthesis.

1. FORMULATION OF THE PROBLEM

Many controlled dynamical systems are described by the following system of differential equations

$$\dot{x} = f(x, u, \upsilon, t) \tag{1.1}$$

Here $x(t)$ is an n-dimensional vector of the phase coordinates, t is the time, $u(t)$ is an m-dimensional vector of the controlling forces, $\upsilon(t)$ is a k-dimensional vector of the disturbances and f is a specified function of its arguments. Systems of the form (1.1) have been investigated in the differential game theory [1]. The disturbance $\upsilon(t)$ can be interpreted as an action of an adversary (the second player), as an uncontrolled external disturbance, and also as an error in the implementation of the control $\upsilon(t)$. The latter case arises if the specified control force (for example, the thrust of an engine) differs from nominal one. In this case the intensity of the possible disturbance $\upsilon(t)$ depends on the value of the control force $u(t)$ and increases as the latter increases.

The problem considered in this paper simulates the situation described above, in which the value of the possible uncontrolled disturbance is due to the error in the control implementation and increases as the value of the control increases. The constraints imposed and the optimality criterion are chosen in such a way that the problem has a solution.

We will now give a formal statement of the problem.

Consider a linear controlled system of the form

$$\dot{x} = A(t)x + B(t)u + C(t)\upsilon + g(t) \tag{1.2}$$

in a specified time interval $t \in [t_0, T]$. Here x, u and υ have the same meaning as in system (1.1), $A(t)$, $B(t)$ and $C(t)$ are specified matrix functions of time of dimension $n \times n$, $n \times m$ and $n \times k$ respectively, and $g(t)$ is a specified n-dimensional vector function of time. A constraint of the following form is imposed on the disturbance $\upsilon(t)$.

$$(G(t)\upsilon(t), \upsilon(t))) \le \|u(t)\|^4 \tag{1.3}$$

Here, $G(t)$ is a specified symmetric positive-definite $k \times k$ matrix function of time, the brackets $(\cdot, \cdot)$ denote the scalar product of vectors, and the symbol $\|\cdot\|$ denotes the Euclidean norm of the vector.

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In particular, if $G(t)$ is the identify matrix, condition (1.3) means that the intensity of possible disturbances is proportional to the square of the value of the control.

The initial condition

$$x(t_0) = x^0 \tag{1.4}$$

is specified at the initial instant $t = t_0$, where x^0 is a specified n-dimensional vector.

The optimality criterion for the control is specified in the form

$$J = \int_{t_0}^{T} [(a(t), x(t)) + b(t)\|u\|^2]dt + F(x(T)) \tag{1.5}$$

Here $a(t)$ is a given n-dimensional vector function of time, $b(t)$ is a given positive scalar function and $F(x)$ is a given continuous scalar function of the vector argument. The optimality criterion J includes an integral term, linear in x and proportional to the projection of the phase vector onto a given direction, a term quadratic in the control u, and also a non-linear terminal term, which can serve as a measure of the deviation of the terminal state $x(T)$ from a specified point. The introduction of this non-linear term is an essential extension of the problem statement considered in [2]. All the given functions of time, which occur in relations (1.2), (1.3) and (1.5), are piecewise-continuous in the interval $t \in [t_0, T]$.

We will formulate the problem of obtaining a control which provides the following minimax to the function J from (1.5)

$$J^* = \min_u \max_v J \tag{1.6}$$

when relations (1.2)–(1.4) are satisfied. Here the maximum is taken with respect to all the disturbances $v(t)$, which satisfy constraint (1.3) under specified control $u(t)$ known over the whole interval $t \in [t_0, T]$. The minimum in (1.6) is taken over all the controls $u(t)$. Here we will construct both an open-loop control $u(t)$, corresponding to the "worst" case of the realization of the disturbance $v(t)$, and also a control in the closed-loop form (the feedback control) as a function $u = u^*(x, t)$ of time and the actual state, calculated for the worst future realization of the disturbance. The problem can be treated as a problem of the control for an ensemble of trajectories of system (1.2), guaranteed (in the sense of optimality criterion (1.6)) for all possible permissible constraints.

2. TRANSFORMATIONS

We will carry out some transformations which simplify the problem. We will denote the fundamental matrix of system (1.2) by $\Phi(t)$. This matrix is defined by the relations

$$\dot{\Phi} = A(t)\Phi, \quad \Phi(t_0) = E_n \tag{2.1}$$

where E_n is the identity $n \times n$ matrix. Assuming the fundamental matrix to be known, we will replace the phase vector and the disturbance in (1.2) as follows:

$$x = \Phi(t)y, \quad v = \|u\|^2 D(t)w \tag{2.2}$$

We will choose the square non-degenerate matrix $D(t)$ of dimension $k \times k$ in such a way that the following identity is satisfied for all $t \in [t_0, T]$

$$D^T G D = E_k \tag{2.3}$$

Relation (2.3) means that the transformation of the disturbance in (2.2) should reduce the quadratic form (1.3) to the sum of squares. One can take the matrix $G^{-1/2}$, for example, as $D(t)$.

After substituting (2.2), system (1.2) takes the form

$$\dot{y} = B_1(t)u + \|u\|^2 C_1(t)w + g_1(t) \tag{2.4}$$

We have introduced the following notation here

$$B_1(t) = \Phi^{-1}(t)B(t), \quad C_1(t) = \Phi^{-1}(t)C(t)D(t)$$
$$g_1(t) = \Phi^{-1}(t)g(t) \tag{2.5}$$

Constraint (1.3), taking (2.2) and (2.3) into account, can be converted to the simple form

$$\|w\| \le 1 \tag{2.6}$$

By making the replacement of variables (2.2), we can write the initial condition (1.4) and the optimality criterion (1.5) in the form

$$y(t_0) = x^0 \tag{2.7}$$

$$J = \int_{t_0}^{T} [(a_1(t), y(t)) + b(t)\|u\|^2]dt + F_1(y(T)) \tag{2.8}$$

Here we have put

$$a_1(t) = \Phi^T(t)a(t), \quad F_1(y) = F(\Phi(T)y) \tag{2.9}$$

Hence, the initial problem (1.2)–(1.6) reduces to finding the minimax

$$J^* = \min_u \max_w J \tag{2.10}$$

for system (2.4) with constraint (2.6) on disturbance w, with initial conditions (2.7) and with optimality criterion (2.8). Using the maximum principle [3], we determine initially the maximum J with respect to w and then the minimum with respect to u.

3. CONSTRUCTION OF THE SOLUTION

For the problem of the maximum of functional (2.8) with respect to w, we will set up a Hamilton function for system (2.4) with optimality criterion (2.8)

$$H = (p, B_1u) + \|u\|^2(C_1^Tp, w) + (p, g_1) + (a_1, y) + b\|u\|^2 \tag{3.1}$$

We write the conjugate system and the transversality condition for the conjugate vector $p(t)$

$$\dot{p} = -a_1(t), \quad p(T) = \partial F_1(y(T))/\partial y \tag{3.2}$$

Solving Cauchy problem (3.2) we obtain

$$p(t, y(T)) = \int_t^T a_1(\tau)d\tau + \frac{\partial F_1(y(T))}{\partial y} \tag{3.3}$$

The condition for a maximum of Hamiltonian (3.1) with respect to w with constraint (2.6) gives

$$w(t) = \|C_1^Tp\|^{-1}C_1^Tp \tag{3.4}$$

Substituting $w(t)$ from (3.4) into system (2.4), we obtain

$$\dot{y} = B_1u + \|u\|^2\|C_1^Tp\|^{-1}C_1C_1^Tp + g_1 \tag{3.5}$$

We will now consider the problem of the minimum of the functional (2.8) with respect to u for system (3.5). The Hamilton function for this optimal-control problem has the form

$$H_1 = (p_1, B_1u) + \|u\|^2\|C_1^Tp\|^{-1}(p_1, C_1C_1^Tp) + (p_1, g_1) - (a_1, y) - b\|u\|^2 \tag{3.6}$$

Here p_1 is the conjugate vector for the problem considered, which satisfies the following conjugate system and transversality condition

$$\dot{p}_1 = a_1(t), \quad p_1(T) = -\partial F_1(y(T))/\partial y \tag{3.7}$$

Comparing relations (3.7) and (3.2), we find that the following equality holds

$$p_1 = -p(t, y(T)), \quad t \in [t_0, T] \tag{3.8}$$

The function $p(t, y(T))$ is defined by (3.3).

Substituting (3.8) into the Hamiltonian (3.6), we obtain, after simplifications

$$H_1 = -(B_1^T p, u) - \|u\|^2(\|C_1^T p\| + b) - (p, g_1) - (a_1, y) \tag{3.9}$$

The maximum of the Hamiltonian H_1 from (3.9) with respect to u is attained at

$$u(t) = -\frac{B_1^T p}{2(\|C_1^T p\| + b)} \tag{3.10}$$

Formulae (3.10) and (3.4) express the optimal open-loop control $u(t)$ and the "worst" disturbance $w(t)$ in terms of the specified functions of time and the conjugate vector p, defined by (3.3) and dependent, in turn, on the terminal value $y(T)$ of the state vector. Hence, to obtain a solution in final form it is necessary to determine $y(T)$ as well.

We substitute the control $u(t)$ from (3.10) into system (3.5) and integrate it with initial condition (2.7). We obtain

$$y(t) = x^0 + \int_{t_0}^{t} L(p, \tau)d\tau \tag{3.11}$$

where we have introduced the notation

$$L(p, t) = \frac{\|B_1^T p\|^2 C_1 C_1^T p}{4\|C_1^T p\|(\|C_1^T p\| + b)^2} - \frac{B_1 B_1^T p}{2(\|C_1^T p\| + b)} + g_1 \tag{3.12}$$

The dependence of the function $L(p, t)$ on the second argument is due to the dependence of the specified function $b(t)$, and also of the functions $B_1(t)$, $C_1(t)$ and $g_1(t)$ on t; the latter functions are defined by (2.5). The function $p(t, y(T))$ is defined by (3.3).

Assuming $t = T$ in (3.11) we obtain

$$y(T) = x^0 + \int_{t_0}^{T} L(p(t, y(T)), t)dt \tag{3.13}$$

Equation (3.13) is a system of n equations for the unknown n-dimensional vector $y(T)$. The question of the existence and uniqueness of the solution of this system remains open in general. Below we consider some special cases in which a unique solution of this equation exists.

Suppose a unique solution $y(T)$ of system (3.13) has been obtained. Then, we arrive at the following procedure for solving the minimax problem. When the vector $y(T)$ is obtained, the conjugate vector $p(t, y(T))$ is completely defined by (3.3), the optimal open-loop control $u(t)$ and the worst disturbance $w(t)$ are specified by Eqs (3.10) and (3.4) respectively, while the state vector $y(t)$ for given $u(t)$ and $w(t)$ is defined in the form of quadrature (3.11). The required minimax J^* of the functional J is obtained by substituting into (2.8) the functions $u(t)$ obtained from (3.10) and $y(t)$ obtained from (3.11) and calculating one more quadrature. The phase trajectory $x(t)$ and the disturbance $\upsilon(t)$ in the initial variables are found using relations (2.2).

We have described above a procedure for constructing the optimal programmed control $u(t)$ for given initial condition $x(t_0) = x^0$ from (1.4). The control obtained, which depends on the initial data, can be represented in the form of the function $u = \tilde{u}(t; t_0, x^0)$. Similarly, the worst disturbance can be represented in the form $\upsilon = \tilde{\upsilon}(t; t_0, x^0)$. In order to obtain the optimal feedback control, i.e. in the form of a synthesis, we must identify the current instant t and the initial instant t_0. The resulting functions

$$\bar{u}(x, t) = \tilde{u}(t; t, x), \quad \bar{\upsilon}(x, t) = \tilde{\upsilon}(t; t, x) \tag{3.14}$$

represent, respectively, the required optimal control and the worst disturbance in the form of a synthesis.

Note, that if the disturbance differs from the worst one, then, using feedback optimal control $\tilde{u}(x, t)$, we gain an advantage in the functional (compared with its value for the open-loop control), since the control employed will be optimal for each actual state considered as the initial state.

4. THE LINEAR CASE

The simplest solution is obtained if the function $F(x)$ in the formula for functional (1.5) is linear, i.e.

$$F(x) = (c, x) \tag{4.1}$$

Here c is a given n-dimensional vector. In this case, the conjugate vector considered previously [2] is completely defined by (3.3) and is independent of $y(T)$. From relations (3.3) and (4.1) we have

$$p(t) = c + \int_t^T a_1(\tau) d\tau \tag{4.2}$$

Since $p(t)$ is independent of $y(T)$, there is no need in this case to solve system (3.13), and all the required quantities are found in closed form. The optimal control $u(t)$ and the worst disturbance $w(t)$ are given by formulae (3.10) and (3.4) respectively, into which we must substitute $p(t)$ from (4.2). Formula (3.11) gives the optimal phase trajectory. In this case the control in the closed-loop form $u(t)$ is identical with the open-loop control. In other words, there is no need to recalculate the optimal control at each instant of time, since it turns out to be independent of the actual state and, consequently, of the form of the disturbance. An example was given in [2], in which the solution for the linear case under consideration was presented in an explicit form.

5. EXAMPLE

As an example we will consider the problem of controlling a second-order system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u + \upsilon \tag{5.1}$$

with the initial conditions

$$x_1(t_0) = x_1^0, \quad x_2(t_0) = x_2^0 \tag{5.2}$$

Here x_1 is the coordinate of the system, x_2 is its velocity, u is the controlling force and υ is a disturbance, which is the error in the control implementation. We will take constraint (1.3) on the disturbance in the form

$$|\upsilon| \le ku^2 \tag{5.3}$$

where $k > 0$ is a given constant.

We will specify the optimality criterion (1.5) in the form

$$J = b\int_{t_0}^T u^2 dt + x_1^2(T) \tag{5.4}$$

where $b > 0$ is a given constant. Criterion (5.4) includes the resource of the control and the quadratic error in bringing the system to zero with respect to the coordinate.

The fundamental matrix $\Phi(t)$ of system (5.1), which satisfies conditions (2.1), and the inverse matrix to it $\Phi^{-1}(t)$ are

$$\Phi(t) = \begin{Vmatrix} 1 & t-t_0 \\ 0 & 1 \end{Vmatrix}, \quad \Phi^{-1}(t) = \begin{Vmatrix} 1 & t_0-t \\ 0 & 1 \end{Vmatrix} \tag{5.5}$$

For the example (5.1)–(5.4) considered we have, in the notation of (1.2), (1.3) and (1.5)

$$B = C = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}, \quad g = 0, \quad G = k^2, \quad a = 0, \quad F(x) = x_1^2 \tag{5.6}$$

The matrices and vectors D, B_1, C_1, g_1 and a_1, defined by relations (2.3), (2.5), (2.9) and (5.6) and the fundamental matrix (5.5), are

$$D = k, \quad B_1(t) = \begin{Vmatrix} t_0 - t \\ 1 \end{Vmatrix}, \quad C_1 = kB_1, \quad g_1 = a_1 = 0 \tag{5.7}$$

Calculating the function $F_1(y)$ and its derivatives with respect to y using (2.9), (5.5) and (5.6), we obtain

$$F_1(y) = F(\Phi(T)y) = Y^2, \quad \frac{\partial F_1(y)}{\partial y} = 2Y\begin{Vmatrix} 1 \\ T - t_0 \end{Vmatrix} \tag{5.8}$$

where

$$Y = y_1 + (T - t_0)y_2 \tag{5.9}$$

The conjugate vector (3.3), by virtue of the last equality for a_1 in (5.7) and (5.8), turns out to be constant

$$p = 2Y(T)\begin{Vmatrix} 1 \\ T - t_0 \end{Vmatrix} \tag{5.10}$$

We substitute the expressions for B_1, C_1 and g_1 from (5.7) and p from (5.10) into formula (3.12) for $L(p, t)$. After simplifications and introducing the notation

$$\eta = Y(T), \quad s = 2k|\eta|b^{-1}, \quad Z = (T - t)s, \quad z = (T - t_0)s \tag{5.11}$$

we will have

$$L(p, t) = \frac{\eta Z(2 + Z)}{2bs(1 + Z)^2}\begin{Vmatrix} t - t_0 \\ -1 \end{Vmatrix} \tag{5.12}$$

We substitute (5.12) into Eq. (3.13) and evaluate the integral. We obtain

$$\int_{t_0}^{t} L d\tau = \frac{\eta}{2bs}\begin{Vmatrix} \dfrac{(t - t_0)^2}{2} - \dfrac{t - t_0}{s(1 + Z)} - \dfrac{1}{s^2}\ln\dfrac{(1 + Z)}{(1 + z)} \\ t_0 - t + \dfrac{1}{s(1 + Z)} - \dfrac{1}{s(1 + z)} \end{Vmatrix} \tag{5.13}$$

We put $t = T$ (and then $Z = 0$) in (5.13) and substitute the expression obtained into (3.13). We have

$$\begin{Vmatrix} y_1(T) \\ y_2(T) \end{Vmatrix} = \begin{Vmatrix} x_1^0 \\ x_2^0 \end{Vmatrix} + \frac{\eta}{2bs^3}\begin{Vmatrix} z^2/2 - z + \ln(1 + z) \\ -sz^2(1 + z)^{-1} \end{Vmatrix} \tag{5.14}$$

Note that the quantities η and s, which occur on the right-hand side of (5.14), are expressed in terms of the vector $y(T)$ by means of Eqs (5.9) and (5.11). Hence, Eq. (5.14) represents a system of two transcendental equations for the components $y_1(T)$ and $y_2(T)$ of the vector $y(T)$.

In order to reduce this system to a single equation we first substitute $y_1(t)$ and $y_2(t)$ from (5.14) into formulae (5.9) and (5.11) for η. By resolving the relation obtained with respect to η and introducing the notation

$$\psi(z) = \frac{2 + z + z^2}{2z(1 + z)} - z^{-2}\ln(1 + z) \tag{5.15}$$

we will have

$$\eta = [1 + (T - t_0)^3(2b)^{-1}z^{-1}\psi(z)]^{-1}[x_1^0 + (T - t_0)x_2^0] \tag{5.16}$$

Taking into account the fact that $\psi(z) > 0$ for all $z \geq 0$, we calculate the modulus of both sides of Eq. (5.16) and substitute the expression obtained for $|\eta|$ into the second and last relations of (5.11). We thereby obtain the following equation for z

$$z + (T - t_0)^3(2b)^{-1}\psi(z) = 2k(T - t_0)b^{-1}\xi \tag{5.17}$$

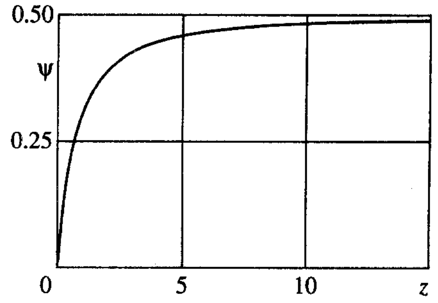

Fig. 1

Here we have put

$$\xi = \left|x_1^0 + (T - t_0)x_2^0\right| \tag{5.18}$$

The function $\psi(z)$ strictly increases when $z > 0$ (see Fig. 1). Hence, the left-hand side of Eq. (5.17) is a monotone function, strictly increasing from 0 to ∞. Consequently, this equation has unique solution $z > 0$ for all values of the positive parameters $T - t_0$, k, b, ξ.

In the limiting cases of small and large values of ξ, using the asymptotic expansions

$$\psi(z) \sim 2z/3 \quad \text{as} \quad z \to 0, \quad \psi(z) \to 1/2 \quad \text{as} \quad z \to \infty \tag{5.19}$$

the solution of Eq. (5.17) can be found in explicit form

$$z = 2k(T - t_0)\xi[b + (T - t_0)^3/3]^{-1} + \dots \quad \text{as} \quad \xi \to 0 \tag{5.20}$$

$$z = 2k(T - t_0)\xi b^{-1} + \dots \quad \text{as} \quad \xi \to \infty \tag{5.21}$$

The solution of Eq. (5.17) is easily obtained numerically in the general case for specified values of the parameters $T - t_0$, k, b, ξ. After this we determine η from formula (5.16), s using the second equality of (5.11), and then the vector $y(T)$ from relation (5.14). The optimal open-loop control $u(t)$ is found from formula (3.10), in which we must substitute the expressions for B_1 and C_1 from (5.7) and for p from (5.1). We finally obtain

$$u(t) = \frac{\eta(t - T)}{2k|\eta|(T - t) + b} \tag{5.22}$$

The quantity η is defined by (5.16). The worst disturbance $\upsilon(t)$ is given by relations (2.2) and (3.4), into which we must substitute expressions (5.7) for D and C_1, expression (5.10) for p and expression (5.22) for $u(t)$. We have

$$\upsilon(t) = \frac{k\eta|\eta|(T - t)^2}{[2k|\eta|(T - t) + b]^2} \tag{5.23}$$

We can determine the optimal trajectory $y(t)$ by substituting into (3.11) the expression for the integral (5.13). After this, using the relations $x(t) = \Phi(t)y(t)$ from (2.2) and equalities (5.5), we obtain the optimal trajectory $x(t)$ in the initial variables. In particular, we obtain (see (5.9) and (5.11))

$$x_1(T) = y_1(T) + (T - t_0)y_2(T) = \eta \tag{5.24}$$

We obtain the minimax value J^* of the functional J by substituting expressions (5.22) and (5.24) into relation (5.4). Expressing η in terms of z from formulae (5.11), we obtain

$$J^* = \frac{b(T - t_0)}{2k^2}\left[\frac{2 + z}{1 + z} - \frac{2}{z}\ln(1 + z)\right] + \frac{b^2 z^2}{4k^2(T - t_0)^2} \tag{5.25}$$

In the limiting cases $\xi \to 0$ and $\xi \to \infty$, using the asymptotic expansions (5.20) and (5.21), we have

$$J^* = \xi^2[1+(T-t_0)^3(3b)^{-1}]^{-1} + \dots \quad \text{as} \quad \xi \to 0$$

$$J^* = \xi^2 + \dots \quad \text{as} \quad \xi \to \infty$$

The quantity ξ is defined by formula (5.18).

We have thus completed the solution of the minimax problem in the open-loop form. To obtain the solution in the closed-loop, or feedback form we carry out the constructions described at the end of Section 3.

We put $t_0 = t$ and $x^0 = x$ in relations (5.11) and (5.16)–(5.18). We then obtain

$$z+(T-t)^3(2b)^{-1}\psi(z) = 2k(T-t)b^{-1}\xi, \quad \xi = |x_1+(T-t)x_2)| \tag{5.26}$$

$$\eta = \frac{x_1+(T-t)x_2}{1+(T-t)^3 2b^{-1}\psi(z)} \tag{5.27}$$

The feedback forms of the optimal control $u = \bar{u}(x, t)$ and of the worst disturbance $\upsilon = \bar{\upsilon}(x, t)$ are determined as follows. Initially, using the actual values of the time t and the phase vector x one must solve transcendental equation (5.26) for z. Having found the unique positive solution z of this equation, we determine η from formula (5.27) and substitute the value obtained into Eqs (5.22) for u and (5.23) for υ. We obtain the required feedback control $\bar{u}(x, t)$ and disturbance $\bar{\upsilon}(x, t)$. We put $t_0 = t$ in (5.25) and obtain the minimax value of the functional as a function of the actual values of t and x.

We will consider a special case when there are no disturbances: $\upsilon = 0$ in (5.1). To do this we put $k = 0$ in relation (5.3). In order to obtain a solution for this case, we take the limit as $k \to 0$ in the relations obtained. We put $z = k\zeta$, where ζ is a new variable. From the asymptotic expansion (5.20) we obtain, as $k \to 0$,

$$\zeta = 2(T-t_0)\xi[b+(T-t_0)^3/3]^{-1} \tag{5.28}$$

Bearing in mind the equality $z = k\zeta$ and also (5.18) and (5.28), from expressions (5.16), (5.22) and (5.25) as $k \to 0$ we obtain

$$\begin{aligned}
\eta &= \frac{x_1^0+(T-t_0)x_2^0}{1+(T-t_0)^3(3b)^{-1}} \\
u(t) &= \frac{\eta(t-T)}{b} = \frac{[x_1^0+(T-t_0)x_2^0](t-T)}{b+(T-t_0)^3/3} \\
J^* &= \frac{b\zeta^2}{4(T-t_0)^2}\left[b+\frac{(T-t_0)^2}{3}\right] = \frac{[x_1^0+(T-t_0)x_2^0]^2}{1+(T-t_0)^3(2b)^{-1}}
\end{aligned} \tag{5.29}$$

It is easy to verify that relations (5.29) define the optimal open-loop control $u(t)$ and the minimum value of the functional for the optimal control problem (5.1)–(5.4) when there are no disturbances. We can obtain the optimal feedback control by replacing t_0 and x_0 by t and x in (5.29).

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